A note on partial content

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Some philosophers have looked for a notion of partial content for which the content of A is in general part of the content of A & B but the content of A ∨ B is not in general part of the content of A.[1] But they have realized that these two requirements are in tension with one another. For A is logically equivalent to (A ∨ B) & A and so, if the content of (A ∨ B) is part of the content of (A ∨ B) & A, it should also be part of the content of A.

There is a related difficulty for allied notions. Thus, one might want A & B to be partially true via A being true though not want A to be partially true via A ∨ B being true (since A ∨ B might be true through B being true, which has nothing to do with A). Or one might want A & B to have at least much truth in it as A even though A does not in general have at least much truth in it as A w B. Or one might want A to confirm A & B but not want A ∨ B to confirm A (since A ∨ B might in its turn be confirmed by B).

In this note, I show that this difficulty is of a quite general nature and does not simply arise from the desire to have the content of A be part of the content of A & B but not have the content of A ∨ B be part of the content of A. For there are three general requirements one might wish to impose on the relation of partial content. The first is that it should hold – or, at least, be capable of holding – in virtue of logical form. Thus, if the content of C is part of the content of A and C′ and A′ are obtained by uniform substitution from C and A, then the content of C′ should also be part of the content of A′. I call this *Substitution*. The second is that whether one statement C is part of the content of another A should be indifferent to their logical formulation. Thus, if C′ is logically equivalent to C and A′ to A then the content of C′ should also be part of the content of A′. I call this *Equivalence*. The third is that the relation should be capable of non-trivial application: we require at least one case in which the content of C is part of the content of A and yet A is not a contradiction, C is not a tautology, and C is a proper logical consequence of A; and we require at least one case in which C is not part of the content of A and yet C is a logical consequence of A. I call this *Non-Triviality*. It is this last requirement which is the generalization of the more particular requirement that the content of A be included in that of A & B even though the content of A ∨ B is not included in the content of A.

It can be shown that when we allow the statements A and C to be formed by means of the usual truth-functional connectives and when logical equivalence is taken to be classical equivalence, that is equivalence according to classical logic, then it is impossible for all three requirements to be met. The proof is in

1 As in Angell (1977), Gemes (1994) and Yablo (2013), for example.

Analysis Vol 0 | Number 0 | 2013 | pp. 1–8 doi:10.1093/analys/ant039

the formal appendix and constitutes a generalization of the informal argument whereby the more particular requirements concerning the partial content of A & B and the partial contenthood of A ∨ B are shown to be in conflict. Thus, no notion of partial content or of confirmation or of being rendered partially true – or whatever – can satisfy all three requirements.

There are a number of possible reactions to this result. One is defeatist. It is to give up on the notion of partial content and perhaps also of the related notions of partial truth and confirmation. This is a response of last resort and I, personally, would be loathe to make it. I feel that philosophers are often far too ready to declare a notion incoherent on general theoretical grounds, without any real appreciation of how problematic those grounds might be.

Another possible reaction is to give up Non-Triviality. But the notion of partial content would then largely lose its interest. For presumably, we only want the content of C to be part of the content of A when it is a logical consequence of A. But suppose that there is no case in which the content of C is not part of the content of A and yet C is a logical consequence of A. Partial content and logical consequence would then coincide and so the notion of partial content would have no independent interest. Suppose now that there were no cases in which the content of C is part of the content of A and yet A is not a contradiction, C is not a tautology and C is a proper logical consequence of A. Then, every case of the content of C being part of the content of A is either one in which A is a contradiction or C is a tautology or A and C are logically equivalent; and so even though partial content and logical consequence need not coincide, there would still be no interesting cases in which the content of C was part of the content of A.

A third reaction is anti-structural. Partial content is not a matter of logical form and so Substitution should be given up. One problem with this response is that it is always possible to guarantee Substitution by stipulation. For suppose, we have a notion of partial content that does not satisfy Substitution. Then, we may define the content of C to be part of the content of A whenever the content of C′ is part of the content of A′, in the original sense of partial content, for any substitution instances C′, A′ of C, A. Partial content is, in effect, partial content in virtue of logical form. And if the notion is redefined in this way, then the response will presumably take the form of rejecting Non-Triviality; there are no non-trivial cases of one statement being a partial content of another in virtue of logical form.

We have already found reason to object to the rejection of Non-Triviality. But our previous objections no longer apply. For even though the defined up structural notion of partial content may not admit of any interesting cases, the original non-structural notion may. We may allow, for example, that the content of p is part of the content of (p & q) for *atomic* statements p and q while denying that the content of p ∨ q is part of the content of p.[2]

2 Gemes (1994, 1997) and Yablo (2013) develop accounts along these lines.

The difficulty with this response is to justify the invidious attitude towards atomic and non-atomic statements. Presumably, if the content of p is part of the content of (p & q) then the content of ~p is part of the content of ~p & ~q and the content of (p & q) is part of the content of (p & q) ∧ r, while the content of (p ∨ q) is not part of the content of (p ∨ q) & p (equivalent to p) and nor is the content of ~(p & q) (equivalent to ~p ∨ ~q) part of the content of ~(p & q) & ~p (equivalent to ~p). But then why is the principle applicable to the negations of atomic statements and to the conjunctions of atomic statements and yet not to the disjunction of atomic statements or even to the negations of their conjunction? What is it that we lose in forming a disjunction or a negation of a conjunction that we do not lose in forming a conjunction or a simple negation?

Presumably, the principle that the content of A is part of the content of A & B will only be true, in general, when the content of A and of B is not 'disjunctive' and the content of ~A will only be non-disjunctive, in general, when the content of A is not 'conjunctive'; and so it is only because we take the content of the atomic statements to be neither conjunctive nor disjunctive that we can allow A to be part of the content of A & B when A and B are atomic statements or their negations or the conjunctions of atomic statements or their negations but not when A and B are the negations of such conjunctions.

This line of defence therefore requires that we distinguish between conjunctive and disjunctive statements, not at the level of language, but at the level of content. I do not want to say that this cannot be done but it raises severe problems and calls for much more in the way of argument than is generally acknowledged.

A more modest form of response along these lines is to hold that there are no non-trivial cases of partial content that hold in virtue of logical form (as with p being part of the content of p & q) but still to allow that there may be non-trivial cases that hold in virtue of 'meaning', as with 'his arm went up' being part of the content of 'he raised his arm'. Part of the problem with this response is that we lose the examples that originally helped motivate the concept of partial content. We wanted to say that the content of A was in general part of the content of A & B but that the content of A ∨ B was not in general part of the content of A. But what now becomes of this motivating idea? This response may not be defeatist; but it is hard to see why someone who makes it should not be defeatist.

A final reaction is anti-classical. We give up the assumption that partial content is preserved under logical equivalence. This assumption has usually been regarded as inviolable. For surely partial content is a matter of truth-conditional content; and surely the truth-conditional content of logical equivalents is the same. I myself think that this objection can be answered, since there is a perfectly natural (perhaps more natural) conception of truth-conditional content under which the truth-conditional content of classical equivalents may not be the same (as argued towards the end of Fine 2012).

But rather than pursuing this line of argument here, let me mention another, more speculative, consideration in favour of the anti-classical response. The judgement that partial content should be preserved under logical equivalence is highly theoretical and derives from adopting a classical conception of content. On the other hand, the judgements that the content of A is part of the content of A & B and that the content of A $\vee$ B is not in general part of the content of A are highly intuitive and are not based on any particular idea of what the content of A & B or of A $\vee$ B should be taken to be. Given a conflict between the theoretical and the intuitive judgements, there is something to be said for going with the intuitive judgements, which are much more likely to lead us to the essential idea of partial content without the disturbance which results from accepting a particular conception of what the content must be. I believe that, once this strategy is pursued, it will lead to a notion of partial content that is subject to Substitution though not to Equivalence and the more classical notion of partial content can then be seen to emerge from the attempt to impose a classical conception of content upon a notion that is more naturally understood against the background of a non-classical conception.

1. *Formal appendix*

Let L be the language of truth-functional logic. We use |- for classical (truth-functional) consequence and -|- for classical equivalence. A formula is said to be *contingent* if both it and its negation are satisfiable. S is said to be a *state description in* the sentence letters $p_1, p_2, \ldots, p_n$, $n \geq 0$, if it is of the form q_1 & q_2 & … & q_n, where each q_i, for $i = 1, 2, \ldots, n$, is either p_i or $\sim p_i$.

We take > (partial content) to be a relation on the formulas of L and consider the following rules on >:

Equivalence (L) $\dfrac{A > C \quad A \text{ -|- } A'}{A' > C}$ (R) $\dfrac{A > C \quad C \text{ -|- } C'}{A' > C}$

Substitution $\dfrac{A' > C}{A' > C' \text{ for } A', C' \text{ a substitution-instance of } A, C}$

Non-Triviality (P) A > C holds for some contingent A and C for which A |- C but not C |- A

(N) A > C fails for some A and C for which A |- C

We read the rules conditionally so that Substitution, for example, says that $A' > C'$ whenever $A > C$.

The proof of impossibility rests on two reductions:

Lemma (First Reduction) Suppose the relation < on L satisfies Equivalence and Substitution. Then, p > (p $\vee$ q) implies A > C whenever A |- C.

Proof Suppose that > satisfies Equivalence and Substitution and that p > (p $\vee$ q). Take any A and C for which A |- C. Then, A > C. For upon substituting A for p and C for q, it follows that $A > (A \vee C)$ by Substitution. But $(A \vee C)$ -|- C; and so $A > C$ by Equivalence.

Lemma (Second Reduction) Suppose that $S_1, \ldots, S_l, S_{l+1}, \ldots, S_m$ are distinct state-descriptions in the sentence letters $p_1, p_2, \ldots, p_n$, with $0 < l < m < 2^n$, and that > is a relation on L satisfying Substitution and Equivalence. Then, $(S_1 \vee \ldots \vee S_l) > (S_1 \vee \ldots \vee S_l \vee S_{l+1} \vee \ldots \vee S_m)$ implies $p > (p \vee q)$.

Proof Without loss of generality, we can assume that the state-description $\sim p_1 \,\&\, \sim p_2 \,\&\ldots\&\, \sim p_n$ is not among the $S_1, \ldots, S_m$, and that, consequently, each of $S_1, \ldots, S_m$ contains at least one un-negated sentence letter $p_1, p_2, \ldots, p_n$ as a conjunct. Pick some distinct sentence letters $w_1, w_2, \ldots, w_m$ and let $v_1, v_2, \ldots, v_m, v_{m+1}$ be the respective formulas $w_1, (\sim w_1 \,\&\, w_2), \ldots, (\sim w_1 \,\&\, \sim w_2 \,\&\ldots\&\, \sim w_{m-1} \,\&\, w_m)$, $(\sim w_1 \,\&\, \sim w_2 \,\&\ldots\&\, \sim w_{m-1} \,\&\, \sim w_m)$. Note that the formulas $v_1, v_2, \ldots, v_{m+1}$ are mutually exclusive and exhaustive.

For each sentence letter p_j, $j = 1, \ldots, n$, we substitute the disjunction of the v_k, $k = 1, 2, \ldots, m$, for which p_j occurs positively (as a conjunct) in S_j. The resulting formulas $S_1', \ldots, S_m'$ can then be simplified (modulo logical equivalence). For since the $v_1, v_2, \ldots, v_m + 1$ are mutually exclusive and exhaustive, $\sim(v_1 \vee v_2)$, for example, can be replaced by $(v_3 \vee v_4 \vee \ldots \vee v_{m+1})$; and similarly for the other cases in which a disjunction of the $v_1, v_2, \ldots, v_m$ is negated. Thus, each conjunct in a state description becomes a (non-empty) disjunction of formulas of the form $v_1, v_2, \ldots, v_{m+1}$. We can now apply the Distribution Law so that each state description becomes a disjunction of conjunctions of formulas of the form $v_1, v_2, \ldots, v_{m+1}$. Since the formulas $v_1, v_2, \ldots, v_{m+1}$ are mutually exclusive, any conjunction containing two of $v_1, v_2, \ldots, v_{m+1}$ is contradictory and can be removed. We therefore end up with a disjunction of formulas from $v_1, v_2, \ldots, v_{m+1}$.

It is clear from the construction that v_j must be one of the disjuncts attached to S_j, $j = 1, 2, \ldots, m$. It should also be clear that v_k cannot be one of the disjuncts attached to S_j for $k \neq j$ since then the conjuncts of S_j and S_k would be the same. It follows that $S_1' \vee \ldots \vee S_l'$ will be equivalent to $v_1 \vee \ldots \vee v_l$, which is equivalent to $w_1 \vee \ldots \vee w_l$, and that $S_1' \vee \ldots \vee S_l' \vee \ldots \vee S_m'$ will be equivalent to $v_1 \vee \ldots \vee v_l \vee \ldots \vee v_m$, which is equivalent to $w_1 \vee \ldots \vee w_l \vee w_{l+1} \vee \ldots \vee w_m$. Substituting p for each of $w_1, \ldots, w_l$ and q for each of $w_{l+1}, \ldots, w_m$, we get $p > p \vee q$.

Theorem (Impossibility) No relation > on L satisfies Equivalence, Substitution and Non-Triviality.

Proof By Positive Non-Triviality, $A > C$ holds for some contingent A and C for which A |- C but not C |- A. Let $p_1, p_2, \ldots, p_n$ be the sentence letters of A and C. Then, each of A and C are equivalent to a disjunction, A′ and C′, of state descriptions in $p_1, p_2, \ldots, p_n$. Since A |- C, we may assume that A′ is of the form $(S_1 \vee \ldots \vee S_l)$ and C′ of the form $(S_1 \vee \ldots \vee S_m)$, for $m \geq l$. Since A and C are contingent, we may assume $l > 0$ and $m < 2n$ and, since not C |- A, we may assume $m > l$. By the Second Reduction, $p > p \vee q$; and by the First Reduction, $A > C$ whenever A |- C, contrary to Negative Non-Triviality.

Theorem (Independence) The conditions Left Equivalence, Right Equivalence, Substitution, Positive Non-Triviality and Negative Non-Triviality are independent.

Proof Let us go through each condition in turn and show how it may fail to be satisfied while the other conditions are satisfied. Verification in each case is left to the reader.

Left Equivalence Let $A > C$ iff A |- C and A is a conjunction.
Right Equivalence Let $A > C$ iff A |- C and C is a conjunction.
Substitution Let $A > C$ iff A |- C but not C|- A.
Positive Non-Triviality Let $A > C$ never hold.
Negative Non-Triviality Let $A > C$ always hold.

The impossibility result is sensitive both to the underlying language L and to the underlying logic on L. It does not hold for classical predicate logic, for example. For take $A > C$ to hold when C is a classical consequence of A and A is a finite consequence of C, that is, A is true in any finite model in which C is true. Then, it may be shown that all three conditions are satisfied. In particular, Non-Triviality holds. For let $\varphi(R)$ be a formula which says that R is a partial strict order without last element. Then, Positive Non-Triviality is satisfied upon letting $A = \varphi(R)$ & p and $C = \varphi(R)$, while Negative Non-Triviality is satisfied upon letting $A = p$ and $C = p \vee q$. Nor does the result hold for intuitionistic sentential logic (or many other subsystems of classical logic). For take $A > C$ to hold when C is an intuitionistic consequence of A and A is a classical consequence of C. Then, Positive Non-Triviality, in particular, is satisfied upon letting $A = p$ and $C = {\sim}{\sim}p$, while Negative Non-Triviality is satisfied, as before, upon letting $A = p$ and $C = p \vee q$. Nor does the result hold for the modal logic T (or any normal extension of it in which the modalities do not collapse). For take $A > C$ to hold when $A \supset \Box C$ is a theorem of T. Then, Positive Non-Triviality is satisfied upon letting $A = \Box p$ and $C = p$ while Negative Non-Triviality is satisfied upon letting $A = C = p$.[3]

3 I owe this last example to the anonymous referee for the journal.

These counterexamples to the impossibility result are not likely to be of an succour to the proponent of partial content, but I do not know of any reasonable extension of the result whereby they might be excluded.[4]

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4 I should like to thank Branden Fitelson and the anonymous referee of the journal for helpful comments.

Abstract

It is shown that certain natural constraints trivialize the concept of partial content and it is suggested, in the light of this difficulty, that the principle that partial content is preserved under the substitution of logical equivalents should be given up.

Keywords: partial content, partial truth, confirmation, non-classical logic, impossibility theorem