Angellic Content

In a number of publications dating from 1977, Angell developed various systems of analytic entailment. The intended interpretation of a statement $A \rightarrow B$ of analytic entailment is that the content of B should be part of the content of A, and a guiding principle behind the understanding of partial content is that the content of A and of B should each be part of the content of $A \wedge B$ but that the content of $A \vee B$ should not in general be part of the content of either A or B. Thus partial content cannot be understood as classical consequence or even as relevant consequence under its more usual interpretation.

Quite independently of Angell's work, I had attempted to develop a semantics for partial content in terms of truthmakers. It was taken to be an intuitive requirement on a truthmaker, or verifier, for a given statement that the verifier should be relevant to the truth of the statement and I had thought that one might take the analytic entailment $A \rightarrow B$ to hold if every verifier for A contained a verifier for B and if every verifier for B was contained in a verifier for A. I was naturally interested in the resulting logic of entailment.

Much to my surprise, I discovered that the resulting logic coincided with the first degree fragment of Angell's system. Under the proposed account of partial content, his system exactly captures the logic of partial content, once the content of a statement is identified with a suitable set of verifiers. The only previously existing semantics for Angell's system of which I am aware is that of Correia [2004]; but his semantics is not at all intuitive and not related in any obvious way to the present semantics. In any case, it is clear that Angell had something like the present semantics in mind. In Angell [1989], p. 123), he writes:

> Ultimately, all justification [of the proposed criterion of analytic equivalence] must rest on a semantic theory of truth-conditions according to which two truth-functional sentential compounds will have the same meaning if and only they have the same set of truth-conditions (not to be confused with 'express the same truth-functions').

Thus the present paper can be regarded as a vindication of this idea.

My focus has been on the first degree fragment of Angell's system. But I believe that the truthmaker approach gives rise to a general framework within which other systems of and other approaches to analytic entailment can be explored. I also believe - along with Angell [2002]. , Gemes [1997], Yablo [2013] and others - that the notion of partial content has a wide range of applications within philosophy and linguistics. I hope to deal elsewhere with the alternatives to Angell's system, with the various applications of the notion, and with the more general philosophical issues raised by the applications and by the choice of one system over another in some sequels to the present paper.

The paper has 10 sections in all. I detail the systems of analytic entailment to be considered (§1). I provide an outline of the truthmaker semantics (§2), give a definition of containment as a relation between contents (§3), and relate containment to the notion of subject-matter (§4). I establish soundness (§5) and then establish completeness by means of disjunctive normal forms (§§6-7). I consider two alternative semantics for the system, one in terms of falsifiers as well as verifiers (§8), and the other in terms of a many-valued logic (§9). I conclude by briefly considering some of the ways in which the system might be extended (§10).[1]

[1] I should like to thank a number of people for helpful discussion of the topic of this paper. They include the participants at a seminar at NYU and the audiences at talks which I gave at the New York Philosophical Logic Group and at a workshop of the Mathematical Philosophical Group in

§1 The Logic of Analytic Containment

We provide our own formulation of Angell's system AC of analytic containment (Angell [1989]). A *propositional* formula is one constructed from the sentence-letters p_1, p_2, ... by means of the connectives ¬, ∧ and ∨; and an *equivalential* formula is one of the form A ↔ B, where A and B are propositional formulas. The system AC provides axioms and rules for proving equivalential formulas.

Our axioms and rules for AC are somewhat more cumbersome than Angell's but they have two advantages. The first is that they break down the various principles into simpler components. This will be helpful later when we wish to consider dropping various of the axioms. The second is that Negation Replacement (A ↔ B / ¬A ↔ ¬B) will be admissible rule of the system. This will later be helpful in developing a 'positive' semantics for the system.

Now for the axioms and rules[2]:

E1 A ↔ ¬¬A
E2 A ↔ A ∧ A
E3 A ∧ B ↔ B ∧ A
E4 (A ∧ B) ∧ C ↔ A ∧ (B ∧ C)
E5 A ↔ A ∨ A
E6 A ∨ B ↔ B ∨ A
E7 (A ∨ B) ∨ C ↔ A ∨ (B ∨ C)
E8 ¬(A ∧ B) ↔ (¬A ∨ ¬B)
E9 ¬(A ∨ B) ↔ (¬A ∧ ¬B)
E10 A ∧ (B ∨ C) ↔ (A ∧ B) ∨ (A ∧ C)
E11 A ∨ (B ∧ C) ↔ (A ∨ B) ∧ (A ∨ C)
E12 A ↔ B / B ↔ A
E13 A ↔ B, B ↔ C / A ↔ C
E14 A ↔ B / (A ∧ C) ↔ (B ∧ C)
E15 A ↔ B / (A ∨ C) ↔ (B ∨ C)

E8 and E9 are De Morgan principles and we call E8 the *first* de Morgan principle and E8 the *second* de Morgan principle. Likewise, E10 and E11 are Distribution principles and we call E10 the *first* Distribution principle and E11 the *second* Distribution principle.

Munich. I have also greatly benefitted from the writings of Fabrice Correia, Lloyd Humberstone, and Steve Yablo.

[2]Branden Fitelson has used a computer program to show that these axioms and rules are indeed independent of one another.

A theorem is any formula of AC that can be obtained from the axioms by means of the rules. We shall often write A -|- B to indicate that A ↔ B is a theorem. The theorems of our system and of Angell's are the same. This will follow easily from corollary 3 below, although we shall not go into the details.

We might also take a connective → for analytic entailment as primitive in place of ↔. A corresponding system may then be obtained by first reading each of the axioms and rules as entailments from left to right and right to left. Thus E1 becomes the two axioms A → ¬¬A and ¬¬A → A. (Certain simplifications may also be made. Only one direction of E3 and E6 are required, E12 may be dropped and E13, E14 and E15 may simply be read with → in place of ↔). We then add A ∧ B → B as the sole special axiom for →. We might call the resulting system $AC_{\rightarrow}$ to distinguish it from the system AC = $AC_{\leftrightarrow}$ in ↔. Proof of the equivalence of the two systems will later follow from the completeness theorem.

Where C(A) is a formula containing a number of occurrences of A, let C(B) be the result of replacing those occurrence with B. We wish to consider the following rules:

<u>Positive Replacement (PR)</u> A ↔ B / C(A) ↔ C(B), as long as the occurrences of A in C(A) are not in the scope of ¬

<u>(Full) Replacement (FR)</u> A ↔ B / C(A) ↔ C(B)

<u>Negation Replacement (NR)</u> A ↔ B / ¬B ↔ ¬A.

An *admissible* rule is one that preserves theoremhood.

<u>Lemma 1</u> PR is a admissible rule.

<u>Proof</u> The result is trivial when C(A) contains no occurrences of A; and so it suffices to prove the result for the case in which C(A) contains one occurrence of A. The proof is by induction.

<u>C(A) = p</u> Trivial

<u>C(A) = (D ∧ E)</u> Suppose C(A) is of the form D(A) ∧ E. By IH, D(A) ↔ D(B) is a theorem. But then D(A) ∧ E ↔ D(B) ∧ E is a theorem by E14. The case in which C(A) is of the form D ∧ E(A) can be reduced to the previous case with the help of E3.

<u>C(A) = (D ∨ E)</u> The proof is similar to the previous case but using E15 and E6 in place of E14 and E3.

Let us note that the proof of this result makes no use of E2 or E11.

<u>Theorem 2</u> NR is an admissible rule.

<u>Proof</u> By induction on the proof of D ↔ E.

E1. D ↔ E is A ↔ ¬¬A and so ¬D ↔ ¬E = ¬A ↔ ¬¬¬A is a theorem by E1.

E2. D ↔ E is A ↔ A ∧ A and ¬D ↔ ¬E is ¬A ↔ ¬(A ∧ A). But:

¬(A ∧ A) -|- ¬A ∨ ¬A by E8
-|- ¬A by E5 and E12.

So ¬(A ∧ A) -|- ¬A by E13; and hence ¬A -|- ¬(A ∧ A) by E12. (Subsequent uses of E12 and E13 will often be implicit).

E3. D ↔ E is A ∧ B ↔ B ∧ A and ¬D ↔ ¬E is ¬(A ∧ B) ↔ ¬(B ∧ A). But:

¬(A ∧ B) -|- ¬A ∨ ¬ B by E8
-|- ¬B ∨ ¬ A by E6
-|- ¬(B ∧ A) by E8 and E12.

E4. D ↔ E is (A ∧ B) ∧ C ↔ A ∧ (B ∧ C) and ¬D ↔ ¬E is ¬[(A ∧ B) ∧ C] ↔ ¬[A ∧ (B ∧ C)]. But:

¬[(A ∧ B) ∧ C] -|- ¬(A ∧ B) ∨ ¬C by E8
-|- (¬A ∨ ¬B) ∨ ¬C by E8 and E15

-|- ¬A ∨ (¬B ∨ ¬C) by E7
-|- ¬A ∨ ¬(B ∧ C) by E8 and E15
-|- ¬(A ∧ (B ∧ C)) by E8.

E5-7. Similar to E2-4.
E8. D ↔ E is ¬(A ∧ B) ↔ (¬A ∨ ¬B) and D ↔ E is ¬¬(A ∧ B) ↔ ¬(¬A ∨ ¬B). But:

¬¬(A ∧ B) -|- A ∧ B by E1
-|- ¬¬A ∧ ¬¬ B by E1, E3 and E14
-|- ¬(¬A ∨ ¬ B) by E9.

E9. Similar to E8.
E10. D ↔ E is A ∧ (B ∨ C) ↔ (A ∧ B) ∨ (A ∧ C) and ¬D ↔ ¬E is ¬[A ∧ (B ∨ C)] ↔ ¬[(A ∧ B) ∨ (A ∧ C)]. But:

¬[A ∧ (B ∨ C)] -|- ¬A ∨ ¬ (B ∨ C) by E8
-|-¬A ∨ (¬B ∧ ¬C) by E9 and E15
-|- (¬A ∨ ¬B) ∧ (¬A ∨ ¬C) by E11
-|- ¬(A ∧ B) ∧ ¬(A ∧ C) by E3, E8 and E14
-|- ¬[(A ∧ B) ∨ (A ∧ C)] by E9.

E11. Similar to E10.
E12. D ↔ E is of the form B ↔ A and obtained from A ↔ B. By IH, ¬A ↔ ¬B is a theorem and so ¬B ↔ ¬A is a theorem by another application of E12.
E13. D ↔ E is of the form A ↔ C and obtained from A ↔ B, B ↔ C. By IH, ¬A ↔ ¬B, ¬B ↔ ¬C are theorems and so ¬A ↔ ¬C is a theorem by another application of E13.
E14. D ↔ E is of the form (A ∧ C) ↔ (B ∧ C) and obtained from A ↔ B. By IH, ¬A ↔ ¬B is a theorem. By E15, ¬A ∨ ¬C ↔ ¬B ∨ ¬C is a theorem and so, by E8, ¬(A ∧ C) ↔ ¬(B ∧ C) is a theorem.
E15. Similar to E14.

<u>Corollary 3</u> Full Replacement is an admissible rule
<u>Proof</u> From the previous two results.

We should note that the previous two results will still hold when E2 is dropped as an axiom.
<u>Lemma 4</u> In the presence of the other principles, the second Distribution principle (E11) is equivalent to NR.
<u>Proof</u> The previous lemma shows that we can derive NP from (E11). We now establish E11 from NP. ¬A ∧ (¬ B ∨ ¬ C) -|- (¬ A ∧ ¬B) ∨ (¬ A ∧ ¬ C) by E10. So ¬[(¬A ∧ (¬ B ∨ ¬ C)] -|- ¬ [(¬ A ∧ ¬B) ∨ (¬ A ∧ ¬ C)] by NP. But:

¬[(¬A ∧ (¬ B ∨ ¬ C)] -|- ¬¬A ∨ ¬(¬B ∨ ¬C) by E8
-|-¬¬A ∨ (¬¬B ∧ ¬¬C) by E9
-|- A ∨ (B ∧ C) by E1

and:

¬ [(¬ A ∧ ¬B) ∨ (¬ A ∧ ¬ C)] -|- ¬(¬ A ∧ ¬B) ∧ ¬(¬ A ∧ ¬ C)] by E9
-|- (¬¬A ∨ ¬¬B) ∧ (¬¬A ∨ ¬¬ C) by E8
-|- (A ∨ B) ∧ (A ∨ C).

It is important to note that NR has been shown to be an *admissible* rule within our system, one that preserves theoremhood. Say that the formula A ↔ B of AC is *derivable from* the set of formulas Δ of AC if it can be obtained from Δ and the axioms of AC by means of the rules of AC. Then, as will later become clear from the semantics, ¬p ↔ ¬q is not derivable from p ↔ q, even though ¬A ↔ ¬B is a theorem whenever A ↔ B is a theorem. It is an interesting question, which we shall not discuss, whether ¬p ↔ ¬q should in general be taken to be inferrable from p ↔ q.

§2 Semantics

Recall that ⊑ is a *partial order* (po) *on S* if it is reflexive, transitive and anti-symmetric relation on *S*. Given the po ⊑ on *S*, we make use of the following standard definitions (with *s*,*t*, *u* ∈ *S* and *T* ⊆ *S*):

s is an *upper bound of T* if *t* ⊑ *s* for each *t* ∈ *T*;

s is a *least upper bound* (lub) *of T* if *s* is an upper bound of *T* and *s* ⊑ *s*′ for any upper bound *s*′ of *T*;

s is *null* if *s* ⊑ *s*′ for each *s*′ ∈ *S* and otherwise is *non-null*;

s ⊏ *t* (*s* is a *proper part of t*) if *s* ⊑ *t* but not *t* ⊑ *s*;

s overlaps t if for some non-null *u*, *u* ⊑ *s* and *u* ⊑ *t*;

s is *disjoint from t* if s does not overlap t.

The least upper bound of *T* ⊆ *S* if it exists is unique (since if *s* and *s*′ are least upper bounds, then *s* ⊑ *s*′ and *s*′ ⊑ *s* and so, by anti-symmetry, *s* = *s*′). We denote it by ⊔*T* and call it the *fusion of T* (or of the members of *T*). When $T = \{t_1, t_2, ...\}$, we shall sometimes write ⊔*T* more perspicuously as $t_1 \sqcup t_2 \sqcup$ ....

We say that the subset *T* of *S* is *complete* if ⊔*T*′ ∈ *S* for any non-empty subset *T*′ of *T*. Note the restriction to non-empty subsets *T*′ and that any non-empty complete subset *T* will have a greatest element ⊔*T*. Given a subset *T* of *S*, there will be a smallest set T^*, the *complete closure of T*, that contains *T* and is complete. It is readily shown that $T^* = \{\sqcup U: U$ a non-empty subset of $T\}$.

A state space S will be a structure of the form (*S*, ⊑) where, intuitively, we think of *S* as the set of states under consideration and of ⊑ as the relation of part-whole on states. More formally, a (*state*) *space* ***S*** is a pair (*S*, ⊑), where *S* is a set and ⊑ is a relation on *S* subject to the following two conditions:

Partial Order (PO) ⊑ is a po on *S*;

Completeness (C) Any subset of *S* has a least upper bound.

Given Completeness, the fusion ⊔*T* of an *arbitrary* subset *T* of *S* will always exist and, in particular, the *null state* ∧ = ⊔∅, which is part of every state, will exist.

A (*state*) *model* ***M*** is an ordered triple (*S*, ⊑, |·|), where (*S*, ⊑) is a state space and and |·| (valuation) is a function taking each sentence letter into a pair (*V*, *F*) of non-empty subsets of *S*. When |p| = (*V*, *F*), we let $|p|^+ = V$ and $|p|^- = F$. We think of $|p|^+$ as the set of verifiers of p and $|p|^-$ as the set of falsifiers of p.

Various other conditions might be imposed on the pairs (V, F) that are assigned to the sentence-letters. It might be required, for example, that the null state not belong to both V and to F. Or we might distinguish a set P of possible states and require that no possible state belong to both V and to F. However, no reasonable condition of this sort will have any impact on the resulting logic in the present context and may safely be ignored.

Given a model $\boldsymbol{M} = (S, \sqsubseteq, |\cdot|)$, we define what it is for an arbitrary formula A to be *verified by* a given state s (s |= A) or *falsified by* the state s (s =| A):

(i)$^+$ s ||- p if $s \in |p|^+$;
(i)$^-$ s -|| p if $s \in |p|^-$;
(ii)$^+$ s ||- ¬B if s -|| B;
(ii)$^-$ s -|| ¬B if s ||- B;
(iii)$^+$ s ||- B ∧ C if for some t and u, t ||- B and u ||- C;
(iii)$^-$ s -|| B ∧ C if s -|| B or s -|| C or s -|| B ∨ C;
(iv)$^+$ s ||- B ∨ C if s |= B or s |= C;
(iv)$^-$ s -|| B ∨ C if for some t and u, t -|| B, u -|| C and $s = t \quad u$.

Clauses of this sort were originally proposed by van Fraassen [1969]. Verification and falsification of this sort will sometimes be called *exact* in order to distinguish it from other notions of verification and falsification that will later be introduced.

We extend the notation $|p|^+$ and $|p|^-$ from sentence letters to arbitrary propositional formulas A and use $|A|^+$ (or often just |A|) for {$s \in S$: s ||- A} and $|A|^-$ for {$s \in S$: s -|| A}. An easy induction establishes:

<u>Lemma 5</u> $|A|^+$ and $|A|^-$ are non-empty subsets of S for each formula A.

It should be noted that this results means that the state space (S, ⊑) should be taken to include 'impossible' states that will verify such formulas as p ∧ ¬p, since a verifier for p and a falsifier for p will fuse to give a verifier for p ∧ ¬p. Fine [2013] defines some natural procedures for constructing impossible states from a state space consisting only of possible states and it turns out to be essential to allow impossible states in providing an adequate semantics for Angell's system.

We shall find it convenient to adopt some more inclusive clauses in place of (iii)$^-$ and (iv)$^+$ (with the other clauses remaining the same):

(iii)$^-$ s -|| B ∧ C if s -|| B or s -|| C or s -|| B ∨ C;
(iv)$^+$ s ||- B ∨ C if s |= B or s |= C or s |= B ∧ C,

or to write them out in full:

(iii)$^-$ s -|| B ∧ C if s -|| B or s -|| C or for some t and u, $s = t \sqcup u$, t -|| B and u -|| C;
(iv)$^+$ s ||- B ∨ C if s |= B or s |= C or for some t and u, $s = t \sqcup u$, t -|| B and u -|| C.

Thus we now take the fusion of falsifiers for B and for C to be a falsifier for B ∧ C and the fusion of verifiers for B and for C to be a verifier for B ∨ C.

There is a systematic connection between the inclusive and non-inclusive form of the semantics. Say that a model $\boldsymbol{M}$ is *complete* (not the same as satisfying the Completeness condition!) if $|p|^+$ and $|p|^-$ are complete for each sentence-letter p. Let $\lceil A \rceil$ = {$s \in S$: s ||- A under the inclusive semantics}. Recall that T^* is the complete closure of T. Then an easy induction establishes:

<u>Lemma 6</u> Suppose the base model $\boldsymbol{M}$ is complete. Then $\lceil A \rceil = |A|^*$.

From henceforth we shall assume that the models under consideration are complete unless there is an explicit indication to the contrary.

§3 Convexity and Containment

Given a state space $\boldsymbol{S} = (S, \sqsubseteq)$, we shall refer to the subsets of S as (*positive*) *contents*. Intuitively, contents constitute the verification (or truth) conditions for a given statement.

Let T (for *top*) and U (for *under*) be two contents of the state space $\boldsymbol{S} = (S, \sqsubseteq)$. We say that T *subsumes* U - in symbols, $T \sqsupseteq U$ - if $(\forall t \in T)(\exists u \in U)[t \sqsupseteq u]$ and that U is *subserves* T - in symbols, $U \sqsubseteq T$ - if $(\forall u \in U)(\exists t \in T)[u \sqsubseteq t]$ (note that $\sqsubseteq$ is not the converse of $\sqsupseteq$). T will be above U iff there is a function f from T into U for which $t \sqsupseteq f(t)$ for each $t \in T$ and U will be below T iff there is a function g from U into T for which $u \sqsubseteq g(u)$ for each $u \in U$. Of course, T may subsume U without U subserving T and vice versa.

We say T *contains* U - in symbols, $T > U$ - if $T \sqsupseteq U$ and $U \sqsubseteq T$. T will contain U iff there is a relation R, with domain T and range U, for which $t \sqsupseteq u$ whenever tRu. Thus under the 'coordinating' relation R, T will contain U just in case it is weakly Pareto superior to U. We might also think, more pictorially, in terms of 'looking down' and 'looking up'; each member of T will look down at a member of U and each member of U will look up at a member of T. (This corresponds to the 'power relations' of Brink ([86]), who puts them to related use).

Containment is the relation between contents which is the analogue of the relation of analytic entailment between statements. Thus we will want to say that A analytically entails C just in case the content of A contains the content of C.

The relation of content containment is intimately related to the property of convexity. Say that a subset T of S is *convex* if $u \in T$ whenever $s, t \in T$ & $s \sqsubseteq u \sqsubseteq t$. In other words, states that lie between member states are also member states. Given a content T, there will be a smallest content T_*, the *convex closure* (or *hull*) *of* T, that contains T and is convex.

The convex closure T_* of T is readily shown to be identical to $\{u \in S$: for some $t_1, t_2 \in T$, $t_1 \sqsubseteq u \sqsubseteq t_2\}$. Indeed, when T is complete, a simpler representation of T_* can be given. For $T \subseteq S$ and $s \in S$, let their *span* $[T, s]$ be $\{u$: $t \sqsubseteq u \sqsubseteq s$ for some $t \in T\}$. Then:

Lemma 7 When T is complete, $T_* = [T, \sqcup T]$ and is itself complete.

Proof Given that T is complete, $\sqcup T \in T$ and so $[T, \sqcup T] \subseteq T_*$. It remains to show that $[T, \sqcup T]$ is complete and convex. So take some $u_1, u_2, ... \in [T, \sqcup T]$. Then for some $s_1, s_2, ... \in T$, $s_i \sqsubseteq u_i \sqsubseteq \sqcup T$ for i = 1, 2 , ... Let $u = \sqcup u_i$. Then $s_1 \sqsubseteq u \sqsubseteq \sqcup T$ and so $u \in [T, \sqcup T]$. Hence $[T, \sqcup T]$ is complete. Now suppose $s \sqsubseteq u \sqsubseteq t$ for $s, t \in [T, \sqcup T]$. Then $t \sqsubseteq \sqcup T$ and $s' \sqsubseteq s$ for some $s' \in T$. But then $s' \sqsubseteq u \sqsubseteq \sqcup T$ and so $u \in [T, \sqcup T]$. Hence $[T, \sqcup T]$ is convex.

When $T = \{t_1, t_2, ...\}$ and $t = \sqcup T$, we might picture their span $[T, \sqcup T]$ by:

t

/ ... \
t_1 t_2 t_3 t_4 ....

Convexity is related to containment in two significant and related ways. First, taking contents to be convex has no effect on the relation of containment::

Lemma 8 $T > U$ iff $T_* > U_*$.

Proof Suppose $T > U$. Take a $u \in T_*$. So $s \sqsubseteq u \sqsubseteq t$ for some $s, t \in T$. Since $T > U$, $s' \sqsubseteq s$ for some $s' \in U$. But then $s' \sqsubseteq u$. Now take $u \in U_*$. So $s \sqsubseteq u \sqsubseteq t$ for some $s, t \in U$. Since $T > U$, $t \sqsubseteq t'$ for some $t' \in T$. But then $u \sqsubseteq t'$.

For the other direction, suppose $T_* > U_*$. Take a $u \in T$ ($\subseteq T_*$). Then for some v, $u \sqsupseteq v$ with $s \sqsubseteq v \sqsubseteq t$ for some $s, t \in U$. But then $u \sqsupseteq t$. Now take a $u \in U$ ($\subseteq U_*$). Then for some v, $u \sqsubseteq v$ with $s \sqsubseteq v \sqsubseteq t$ for $s, t \in T$. But then $u \sqsubseteq t$.

Second, the relation of containment for convex contents is antisymmetric. Write $T <> U$ for $T > U$ and $U > T$. Then:

Lemma 9 Suppose T and U are convex contents. Then $T = U$ if $T <> U$.

Proof Suppose that $T <> U$ and that $s \in T$. Since $U > T$, $s \sqsubseteq s^+$ for some $s^+ \in U$ and, since $T > U$, $s \sqsupseteq s'$ for some $s' \in U$. So $s' \sqsubseteq s \sqsubseteq s^+$ and, given that U is convex, $s \in U$. The other direction is similar.

The second result does not hold for arbitrary complete contents (even as given by the inclusive semantics). For consider the formulas $p \vee (p \wedge q) \vee (p \wedge q \wedge r)$ and $p \vee (p \wedge r) \vee (p \wedge q \wedge r)$ and let T and U be their respective complete contents in some model. Then it is readily verified that $T <> U$. But it will not in general be true that $T = U$, since a verifier for $(p \wedge q)$ and hence for $p \vee (p \wedge q) \vee (p \wedge q \wedge r)$ may not be a verifier for $p \vee (p \wedge r) \vee (p \wedge q \wedge r)$.

If the relation $T < U$ is genuinely to represent a relation of partial content, of T being *part* of the content U, then we would expect the relation to be antisymmetric. Thus what these results show is that the notion of content with respect to which $<$ is a relation of partial content is the notion of convex content. Differences not reflected by the convex closures of T and U are irrelevant to whether the one content is part of the other. In particular, the exact semantics from the previous sections provides more information than is strictly necessary for determining partial content; it is the convex closure of the set of exact verifiers, rather than the set of exact verifiers itself, that should be taken to constitute the content of a statement for the purposes of ascertaining whether one statement is analytically entailed by another. (It is perhaps interesting to note, in this regard, that Gardenfors [2004] takes convexity to be a significant characteristic feature of natural concepts.)

§4 Subject-matter

We may distinguish three different, successively broader, notions of content. Given a propositional formula A, its *exact* content $|A|$ is $\{s \in S: s \Vdash A$ under the non-inclusive semantics$\}$, its *complete* content $\lceil A \rceil$ is the complete closure $|A|^*$ of its exact content, which is the same as $\{s \in S: s \Vdash A$ under the inclusive semantics$\}$, and what we might call its *replete* content [A] is the convex closure $\lceil A \rceil_*$ of its complete content. We might say that $|A|$ consists of the states that *exactly* verify A, $\lceil A \rceil$ of the states that *inclusively* verify A, and [A] of the states that *super-inclusively* verify A.

We previously provided a direct semantic characterization of the complete content $|A|^*$ as the set of verifiers for A under the inclusive semantics. Something similar can be done in the case of the replete content, although in this case it is somewhat more perspicuous to define an intermediate notion of *subject-matter*. Intuitively, the subject-matter of a statement is what it is about and it may be identified with the closure <A> of the set of verifiers of the statement under fusions and parts. When the content of a statement is complete, there will be a maximal verifier

$\sigma(A) = \bigsqcup(\lceil A \rceil) = \bigsqcup(|A|)$ of the statement and the closure <A> will then be identical to $\{s \in S: s \sqsubseteq \sigma(A))$. Thus, in this particular case, the subject-matter may simply be identified with $\sigma(A)$, with the understanding that the statement A will be about $\sigma(A)$ and any of its parts.

This is a 'fact'-based conception of subject-matter; the subject-matter of a statement is given, in effect, by those parts of a possible world which the statement is about. This may or may not be the most general conception of subject-matter (see Lewis [1988] and Yablo [2013] for further discussion), but it is the conception of subject-matter that is most appropriate for our present purposes in explaining the connection with analytic entailment.

The subject-matter $\sigma(A)$ has been defined in terms of the complete content $\lceil A \rceil$. But a direct recursive definition is also possible. We distinguish between the *positive* subject-matter of $\sigma^+(A)$ of A and its *negative* subject-matter $\sigma^-(A)$, which may be identified with the positive subject-matter of $\neg A$:

(i)$^+$ $\sigma^+(p) = \bigsqcup|p|^+$

(i)$^-$ $\sigma^-(p) = \bigsqcup|p|^-$
(ii)$^+$ $\sigma^+(\neg B) = \sigma^-(B)$
(ii)$^-$ $\sigma^-(\neg B) = \sigma^+(B)$
(iii)$^+$ $\sigma^+(B \wedge C) = \sigma^+(B) \sqcup \sigma^+(C)$
(iii)$^-$ $\sigma^-(B \wedge C) = \sigma^-(B) \sqcup \sigma^-(C)$
(iv)$^+$ $\sigma^+(B \vee C) = \sigma^+(B) \sqcup \sigma^+(C)$
(iv)$^-$ $\sigma^-(B \vee C) = \sigma^-(B) \sqcup \sigma^-(C)$

An easy induction shows:

<u>Lemma 10</u> $\sigma^+(A) = \bigsqcup(|A|)$ and $\sigma^-(A) = \bigsqcup(|\neg A|)$.

It follows from these definitions that $\sigma^+(B \wedge C) = \sigma^+(B \vee C)$ and $\sigma^-(B \wedge C) = \sigma^-(B \vee C)$. However, there is no reason in general why $\sigma^+(A)$ should be the same as $\sigma^-(A) = \sigma^+(\neg A)$. We could define the *bi-lateral subject-matter* $\sigma(A)$ *of* A to be $\sigma^+(A) \sqcup \sigma^-(B)\}$ and we would then have $\sigma(A) = \sigma^+(\neg A)$ in addition to the identities $\sigma(B \wedge C) = \sigma(B \vee C) = \sigma(B) \sqcup \sigma(CC)$.

We may show:

<u>Theorem 11</u> $[A] = [|A|, \sigma^+(A)]$ and $[\neg A] = [|A|^-, \sigma^-(A)]$.

<u>Proof</u> $[A] = \lceil A \rceil_*$ by definition

$= [\lceil A \rceil, \sqcup(\lceil A \rceil)]$ since $\lceil A \rceil$ is complete by lemma 5

$= [\lceil A \rceil, \sqcup(|A|)]$ since $\sqcup(|A|) = \sqcup(|A|^*)| = \sqcup(\lceil A \rceil)$

$= [|A|, \sqcup(|A|)]$ since $|A| \subseteq \lceil A \rceil$ and each $s \in \lceil A \rceil$ contains a $t \in |A|$

$= [|A|, \sigma^+(A)]$ since $\sqcup(|A|) = \sigma^+(A)$ by lemma 10.

The proof for [¬A] is similar.

Thus the verifiers of A under the 'replete' semantics are those states 'big' enough to contain an exact verifier but 'small' enough to be included within the subject-matter of the statement.

Given the above result, we can now provide a direct semantics for replete content. We first give a recursive definition of σ^+ and σ^- as above. We then give a recursive specification of the positive and negative replete contents $[A]^+$ and $[A]^-$ in terms of σ^+ and σ^-. Thus, in the case of the positive content for conjunctions and disjunctions, for example, we will have the following two clauses:

$[B \wedge C]^+ = [[B]^+ \sqcup [C]^+, \sigma^+(B \wedge C)]$,

$[B \vee C]^+ = [[B]^+ \cup [C]^+, \sigma^+(B \vee C)]$

(where $[B]^+ \sqcup [C]^+ = \{s \sqcup t: s \in [B]^+ \text{ and } t \in [C]^+\}$).

The relation of containment among the contents [A] and [C] is particularly perspicuous when the contents of A and C are represented in the form $[|A|, \sigma^+(A)]$, $[|C|, \sigma^+(C)]$. For we may then say:

$[|A|, \sigma^+(A)] > [|C|, \sigma^+(C)]$ if (i) $|A| \sqsupseteq |C|$ and (ii) $\sigma^+(C) \sqsubseteq \sigma^+(A)$.

Here (i) corresponds to the first clause of the original definition of containment and specifies an entailment-type relation between the exact verifiers of A and C, while (ii) corresponds to the second clause of the original definition and specifies a relation of inclusion between the subject-matters of C and A. There has been a long tradition - going back to Parry [1933] and re-emerging in the recent work of Yablo[2013] - of thinking of the relation of analytic entailment between A and C as the conjunction of a relation of non-analytic entailment relation between A and C, on the one hand, and a relation of inclusion between the subject matters of C and A, on the other. We see from the above definition how the present account of analytic entailment also conforms - in what is perhaps an especially pleasing way - to this tradition.

§5 Soundness of AC

We say that the formula A ↔ B *holds in* the model ***M*** if [A] = [B], i.e. if their replete content is the same. Spelling out directly what this means in terms of the inclusive semantics, A ↔ B will hold in a model ***M*** if (i) every verifier of A contains and is contained in a verifier of B and (ii) if every verifier of B contains and is contained in a verifier of A. We say that the formula A ↔ B is *valid* if it holds in every model and that the inference $A_1, A_2, ..., A_n/B$, $n \geq 0$, is valid if B holds in any model in which all of $A_1, A_2, ..., A_n$ hold.

If we were to introduce the connective → for analytic entailment into the language, then we might say that A → B holds in a model ***M*** if $\lceil A \rceil > \lceil B \rceil$ or, spelling this out directly, if (i) $\lceil A \rceil \sqsupseteq$

$\lceil B \rceil$, i.e. every verifier of A (under the inclusive semantics) contains a verifier of B and (ii) $\lceil B \rceil \sqsubseteq \lceil A \rceil$, i.e. every verifier of B is contained in a verifier of A. It is worth noting that:

$(A \vee B) \rightarrow B$ holds in a model ***M*** iff $\lceil A \rceil \sqsupseteq \lceil B \rceil$; and

$A \rightarrow (A \vee B)$ holds in a model ***M*** iff $\lceil B \rceil \sqsubseteq \lceil A \rceil$.

Thus the two component clauses (i) and (ii) above can be recovered from $\rightarrow$ and hence from $\leftrightarrow$; and this will mean, in its turn, that the logic for the connectives corresponding to the separate clauses (to be discussed later) can be recovered, via these definitions, from Angell's logic.

The following basic result will be helpful in what follows:

<u>Lemma 12</u> $\lceil A \wedge B \rceil > \lceil A \rceil$ (relative to any model ***M***).

<u>Proof</u> Suppose *s* is a verifier (in ***M***) for $A \wedge B$ under the $\lceil A \wedge B \rceil <> \lceil B \rceil$ semantics. Then it is of the form $t \sqcup u$ where *t* is a verifier for A and *u* a verifier for B; and so *s* contains *t*. For the other direction, suppose *s* is a verifier for A. By lemma 5, there is a verifier *t* for B. But then $u = s \sqcup t$ is a verifier for $A \wedge B$ and contains *t*.

Note the essential use of lemma 5 and of the completeness assumption in the proof of this result.

Within the context of the inclusive semantics, we may define analytic entailment in terms of analytic equivalence, via the definition $A \rightarrow B =_{df} A \wedge B \leftrightarrow A$:

<u>Lemma 13</u> $A \wedge B \leftrightarrow A$ holds in a model iff $A \rightarrow B$ holds in the model.

<u>Proof</u> Assume $A \wedge B \leftrightarrow A$ holds in the model, i.e. $\lceil A \wedge B \rceil <> |A\rceil$. Suppose *s* ||- A (under the inclusive semantics). Then *s* contains a verifier of $A \wedge B$ and therefore contains a verifier of B. Now suppose *s* ||- B. There is a verifier *t* of A; and so $t \sqcup s$ is a verifier of $A \wedge B$. But then $t \sqcup s$, and hence *s*, is included in a verifier of A.

For the other direction, assume $A \rightarrow B$ holds in the model, i.e. $\lceil A \rceil > \lceil B \rceil$. By lemma 12, $\lceil A \wedge B \rceil > \lceil A \rceil$ and so it remains to show $\lceil A \rceil > \lceil A \wedge B \rceil$. Take a verifier *s* of A. Then it contains a verifier *t* of B and so $s = s \sqcup t$ is a verifier of $A \wedge B$. Now take a verifier *s* of $A \wedge B$. Then it is of the form $t \sqcup u$, where *t* is a verifier of A and *u* a verifier of B. But *u* is included in a verifier u^+ of A and so $s = t \sqcup u$ is included in a verifier $t \sqcup u^+$ of A as well.

We now have:

<u>Theorem 14 (Soundness)</u> Every theorem of AC is valid.

<u>Proof</u> We show that each of the axioms and rules is valid. Let us examine a few representative cases. In the proofs below, ||- and -|| are understood to be in conformity with the non-inclusive semantics and ||-* and -||* to be in conformity with the inclusive semantics.

<u>AC1</u> $A \leftrightarrow \neg\neg A$. *s* ||- $\neg\neg A$ iff *s* -|| $\neg A$ iff *s* ||- A. Thus $|\neg\neg A| = |A|$; so $\lceil \neg\neg A \rceil = \lceil A \rceil$; and so $[\neg\neg A] = [A]$, as required.

<u>AC2</u> A ↔ A ∧ A. If *s* ||- A then *s* ⊔ *s* = *s* ||- A ∧ A; and if *s* ||- A ∧ A, then *s* is of the form *t* ⊔ *u* where *t* ||- A and *u* ||- A and so *s* ||-* A ∧ A. Thus |A| ⊆ |A ∧ A| and |A ∧ A| ⊆ ⌈A⌉; so ⌈A⌉ = ⌈A ∧ A⌉; and hence [A] = [A ∧ A].

<u>AC8</u> ¬(A ∧ B) ↔ (¬A ∨ ¬B).

s ||- ¬(A ∧ B) iff *s* -|| A ∧ B
iff *s* -|| A or *s* -|| B
iff *s* ||- ¬A or *s* || ¬B
iff *s* ||- ¬A ∨ ¬B.

Thus |¬(A ∧ B)| = |(¬A ∨ ¬B)|; and so [¬(A ∧ B)] = [(¬A ∨ ¬B)].

<u>AC10</u> A ∧ (B ∨ C) ↔ (A ∧ B) ∨ (A ∧ C).

s ||- A ∧ (B ∨ C) iff *s* = *t* ⊔ *u* for *t* ||- A and *u* ||- B ∨ C
iff *s* =*t* ⊔ *u* for *t* ||- A and (*u* ||- B or *u* ||- C)
iff *s* ||- A ∧ B or *s* ||- A ∧ C
iff *s* ||- (A ∧ B) ∨ (A ∧ C).

Thus |A ∧ (B ∨ C)| = |(A ∧ B) ∨ (A ∧ C)|; and so [A ∧ (B ∨ C)] = [(A ∧ B) ∨ (A ∧ C)].

<u>AC11</u> A ∨ (B ∧ C) ↔ (A ∨ B) ∧ (A ∨ C). Suppose *s* ||- A ∨ (B ∧ C). Then *s* ||- A or *s* ||- B ∧ C. In the first case, *s* ||- (A ∨ B) and *s* ||- (B ∨ C) and so *s* = *s* ⊔ *s* ||- (A ∨ B) ∧ (A ∨ C). In the second case, *s* = *t* ⊔ *u* for *t* ||- B and *u* ||- C. So *t* ||- (A ∨ B) and *u* ||- (A ∨ C) and consequently *s* = *t* ⊔ *u* ||- (A ∨ B) ∧ (A ∨ C). Thus |A ∨ (B ∧ C)| ⊆ |(A ∨ B) ∧ (A ∨ C)| and so:

(*) [A ∨ (B ∧ C)] ⊆ [(A ∨ B) ∧ (A ∨ C)].

Now suppose *s* ||- (A ∨ B) ∧ (A ∨ C). Then *s* = *t* ⊔ *u* for *t* ||- (A ∨ B) and *u* ||- (A ∨ C). We distinguish four subcases:

(a) *t* ||- A and *u* ||- A. Then *s* = *t* ⊔ *u* ||-* A; so *s* ||-* (A ∨ B) and *s* ||-* (A ∨ C); and so *s* ||-* (A ∨ B) ∧ (A ∨ C). Thus |(A ∨ B) ∧ (A ∨ C)| ⊆ ⌈A ∨ (B ∧ C)⌉ and so |(A ∨ B) ∧ (A ∨ C)| ⊆ [A ∨ (B ∧ C)].

(b) *t* ||- A and *u* ||- C. There is a *u′* ||- B. So *u′* ⊔ *u* ||- B ∧ C and *t* ⊔ *u′* ⊔ *u* ||-* A ∨ (B ∧ C). But *t* ⊑ *t* ⊔ *u* ⊑ *t* ⊔ *u′* ⊔ *u*, *t* ||-* A ∨ (B ∧ C) and *t* ⊔ *u′* ⊔ *u* ∈ ⌈A ∨ (B ∧ C)⌉ and so *s* = *t* ⊔ *u* ∈ [A ∨ (B ∧ C)]. Thus again, |(A ∨ B) ∧ (A ∨ C)| ⊆ [A ∨ (B ∧ C)].

(c) *t* ||- B and *u* ||- A. Similar to case (b).

(d) *t* ||- B and *u* ||- C. Then *s* = *t* ⊔ *u* ||- (B ∧ C) and so *s* ||- A ∨ (B ∧ C). Thus again, |(A ∨ B) ∧ (A ∨ C)| ⊆ [A ∨ (B ∧ C)].

In each subcase, |(A ∨ B) ∧ (A ∨ C)| ⊆ [A ∨ (B ∧ C)] and hence:

(**) [(A ∨ B) ∧ (A ∨ C)] ⊆ [A ∨ (B ∧ C)]

From (*) and (**), we obtain [A ∨ (B ∧ C)] = [(A ∨ B) ∧ (A ∨ C)], as required.

E13 A ↔ B, B ↔ C / A ↔ C. Assume A ↔ B, B ↔ C hold in the given model. Then [A] = [B] and [B] = [C]; so [A] = [C]; and so A ↔ C also holds in the model.

AC14 A ↔ B /(A ∧ C) ↔ (B ∧ C). Assume ⌈A⌉ <> ⌈B⌉. To show ⌈A ∧ C⌉ <> ⌈B ∧ C⌉. Suppose *s* ∈ ⌈A ∧ C⌉, i.e. *s* ||-* A ∧ C. Then for some *t* and *u*, *s* = *t* ⊔ *u*, *t* ||- B and *u* ||- C. Since ⌈A⌉ > ⌈B⌉, *t* ⊒

t' for some $t' \in \lceil B \rceil$. But then $s = t \sqcup u \sqsupseteq t' \sqcup u \in \lceil (B \wedge C) \rceil$. We may show similarly that for some $t^+ \in \lceil B \rceil$, $s = t \sqcup u \sqsubseteq t^+ \sqcup u \in \lceil (B \wedge C) \rceil$; and the other direction follows by symmetry.

Note that in the above proof it is only the right to left directions of AC2 and AC11 that require appeal to inclusive as opposed to non-inclusive content.

Finally, let us show how the rule NR is not valid even though it is an admissible rule. We let $\boldsymbol{M} = (S, \sqsubseteq, |\cdot|)$, where $S = \wp(\{0, 1, 2\})$, $\sqsubseteq = \{(s, t): s, t \in S \text{ and } s \subseteq t\}$, $|p| = (\{\{0\}\}, \{\{1\}\})$ and $|q| = (\{\{0\}\}, \{\{2\}\})$. It is then verified that p ↔ q holds in $\boldsymbol{M}$ while ¬p ↔ ¬q does not.

§6 Disjunctive Normal Forms

We shall prove completeness of the system by way of disjunctive normal forms. This is perhaps the simplest way to prove completeness though it is not altogether ideal, since it does not apply to various natural extensions of the system.

A *literal* is a sentence letter or its negation; a *conjunctive form* is a literal or a conjunction of literals (i.e. it is a formula formed from literals by the means of ∧); and a *disjunctive form* (sometimes called a disjunctive normal form) is a conjunctive form or a disjunction of conjunctive forms. It is readily seen that in a disjunctive form no connective will occur within the scope of any occurrence of ¬ and no connective other than ∧ will occur within the scope of any occurrence of ∧. We shall think of a conjunctive form that is identical to a literal as having that literal as its sole conjunct and of a disjunctive form that is identical to a conjunctive form as having that conjunctive form as its sole disjunct.

Say that the formulas A and B are *provably equivalent* (*in* AC) if A ↔ B is a theorem (of AC).

Theorem 15 Any formula is provable equivalent to one in disjunctive normal form.

Proof In the standard way, using the De Morgan Equivalences (E8,9), Double Negation (E1), Distribution of ∧ over ∨ (E10), Positive Replacement (PR), Symmetry (E12) and Transitivity (E13).

Note that the above proof makes no use of Idempotence (E2) or of Distribution of ∨ over ∧ (E11).

We now present some more refined versions of the disjunctive normal form theorem. Let us suppose that the sentence letters of our language are listed in a fixed enumeration (without repeats). A conjunctive form is then said to be *standard* if its conjuncts appear in the fixed order without repeats and with association to the left. There is then only one standard conjunctive form containing given literals as conjuncts. We can also suppose that the standard conjunctive forms are listed in a fixed enumeration. A disjunctive form is then *standard* if each of its disjuncts is standard and if the disjuncts appear in the fixed order and with association to the left. Again, there is only one standard disjunctive form containing given standard conjunctive forms as disjuncts. Thus, a standard disjunctive form is essentially given by a set of sets of literals, with the member sets of literals corresponding to standard conjunctive forms and with the set itself corresponding to a standard disjunctive form of these conjunctive forms.

Given a conjunctive form A, let us use L(A) for the literals that occur (as conjuncts) in A. Let us say that one conjunctive form A *effectively contains* another B if L(A) ⊇ L(B), that two conjunctive forms A and B are *effectively identical* if L(A) = L(B), that two disjunctive forms are

effectively identical if any disjunct of one is effectively the same as a disjunct of the other, and that two disjunctive forms are *strictly equivalent* if they are provably equivalent and effectively identical.
Lemma 16 Each disjunctive form is strictly equivalent to a standard disjunctive form.
Proof We may prove the result in the standard way by appeal to the commutativity, associativity and idempotence axioms for ∧ and ∨ (E2 - 7).

A disjunctive form A is said to be *maximal* if, whenever it contains a disjunct D and literal q (appearing as conjunct of a disjunct), then it contains a disjunct whose literals are exactly those of D along with q (maximality is a feature of disjunctive forms corresponding to repleteness as a feature of content and also to the maximality of conjunctive normal forms as defined in Angell [1989], 126-7).
Lemma 17 Each disjunctive form A is provably equivalent to a maximal disjunctive form.
Proof Let us first note that:

(*) A ∨ (B ∧ C) is provably equivalent to A ∨ (A ∧ B) ∨ (A ∧ C) ∨ (BvC).

For A ∨ (B ∧ C) is provably equivalent to (A ∨ B) ∧ (A ∨ C), by Distribution, which is provably equivalent to (A ∧ A) ∨ (A ∧ C) ∨ (B ∧ A) ∨ (B ∧ C) by Distribution again, which simplifies to A ∨ (A ∧ B) ∨ (A ∧ C) ∨ (B ∧ C).

Let A# be a maximal form of A (clearly, a maximal form always exist). Let A’s *fall-off* value n be the number of times there is no disjunct in A effectively identical to $D \wedge q_1 \wedge q_2 \wedge \ldots q_n$, n > 0, where D is a disjunct of A and $q_1 \wedge q_2 \wedge \ldots q_n$ are literals occurring in D (where all that concerns us is the composition L(D) of a disjunct) . The proof is by induction on n. If n = 0, then A is already maximal. So suppose n > 0. Then there is a disjunct D of A and a literal q of A for which D ∧ q is not also a disjunct of A. Without loss of generality, we can assume that A is of the form D ∨ (q ∧ C) ∨ E, where E is itself a disjunctive form. By (*) above, D ∨ (q ∧ C) is provably equivalent to D ∨ (D ∧ q) ∨ (D ∧ C) ∨ (q ∧ C) and so D ∨ (q ∧ C) ∨ E is provably equivalent to D ∨ (D ∧ q) ∨ (D ∧ C) ∨ (q ∧ C) ∨ E, which has a smaller fall-off value than D ∨ (q ∧ C) ∨ E.

Putting together the three previous results, we obtain:
Theorem 17 (Normal Form) Each formula A is provably equivalent to a standard maximal disjunctive form B.
Proof By theorem 15, A is provably equivalent to a disjunctive form A′; by lemma 17, A′ is provably equivalent to a maximal disjunctive form A″; by lemma 16, A″ is strictly equivalent to a standard disjunctive form B, which must also be maximal given that it is effectively identical to A″.

§7 Completeness of AC

We use an obvious form of canonical model to connect disjunctive forms to the semantics.

The *canonical model* $\boldsymbol{M}_c$ is the triple (S_c, $\sqsubseteq_c$, $|\cdot|_c$), where:

(i) S_c is the set of all sets of literals;
(ii) $\sqsubseteq_c$ is the subset relation restricted to *S*;
(iii) $|p|_c = \{(V, F): V = \{\{p\}\}$ and $F = \{\{\neg p\}\}\}$.

It is readily verified that $\boldsymbol{M}_c$ is indeed a model.

The following results are readily verified:

<u>Lemma 18</u> (i) For any conjunctive form A, $|A|_c = \{L(A)\}$, and

(ii) For any disjunctive form $D = D_1 \vee D_2 \vee ... D_n$, $|D|_c = \{L(D_1), L(D_2), ..., L(D_n)\}$.

(iii) For any maximal disjunctive form D, $|D|_c$ is a replete content.

<u>Lemma 19</u> Let A and B be two standard maximal disjunctive forms. Then:

$[A]_c = [B]_c$ iff A = B.

<u>Proof</u> Suppose that $[A]_c = [B]_c$ and that $A = A_1 \vee A_2 \vee ... \vee A_m$ and $B = B_1 \vee B_2 \vee ... \vee B_n$. Given that A and B are maximal disjunctive forms, $|A|_c$ and $|B|_c$ are replete contents by lemma 18(iii) and so $[A]_c = |A|_c$ and $[B]_c = |B|_c$. By lemma 18(ii), $|A|_c = \{L(A_1), L(A_2), ..., L(A_m)\}$ and $|B|_c = \{L(B_1), L(B_2), ..., L(B_n)\}$ and so $\{L(A_1), L(A_2), ..., L(A_m)\} = \{L(B_1), L(B_2), ..., L(B_n)\}$. But given that A and B are in standard disjunctive form, this implies that A = B.

<u>Theorem 20 (Completeness)</u> Any valid formula $A \leftrightarrow B$ is a theorem of AC.
<u>Proof</u> By the normal form theorem (theorem 17), A is provably equivalent to a standard maximal disjunctive form A′ and B is provably equivalent to a standard maximal disjunctive form B′. By soundness of AC (theorem 14), $A' \leftrightarrow B'$ is also valid and so $[A']_c = [B']_c$. By lemma 19, A′ = B′ and, since A is provably equivalent to A′ and B to B′, A is provably equivalent to B.

The previous results provide a semantic proof of the ‘uniqueness’ theorem of Angell [1989], p. 128:

<u>Theorem 21 (Uniqueness)</u> Each formula A is provably equivalent to a unique standard maximal disjunctive form.
<u>Proof</u> Suppose A is provably equivalent to the standard maximal disjunctive forms A′ and A″. Then A′ is provably equivalent to A″. By Soundness, $[A']_c = [A'']_c$ and so, by lemma 19, A′ = A″.

We may see this result as a concrete ‘syntactic’ version of lemma 9. We may also establish the completeness of the system $AC_{\rightarrow}$, as defined in §1:

<u>Theorem 22</u> Any valid formula $A \rightarrow B$ is a theorem of $AC_{\rightarrow}$.
<u>Proof</u> Suppose $A \rightarrow B$ is valid. Then $A \leftrightarrow A \wedge B$ is valid by lemma 13. By the completeness of AC, $A \leftrightarrow A \wedge B$ is a theorem of AC and hence $A \rightarrow A \wedge B$ is a theorem ofv. But $A \wedge B \rightarrow B$ is a theorem of $AC_{\rightarrow}$ and so, by the →-version of E13, so is $A \rightarrow B$.

We should mention that the canonical model can be confined to the sentence letters occurring in A and B and we may thereby establish that the system AC has the finite model property and is decidable.

<u>§8 The Bi-lateral Semantics for AC</u>

The above semantics for analytic entailment and equivalence has been uni-lateral; it simply relates to the positive content of the formulas in question. We may also provide a bi-lateral semantics for these notions, which relates in a similar way to both the positive and the negative content of the formulas.

In the case of analytic equivalence, we will now say that $A \leftrightarrow B$ *holds in* the model ***M*** if $[A]^+ = [B]^+$ and $[A]^- = [B]^-$ or, to put it in terms of the inclusive semantics, if (i) every verifier (or falsifier) of A contains and is contained in a verifier (or falsifier) of B and (ii) if every verifier (or falsifier) of B contains and is contained in a verifier (or falsifier) of A.

We might read off the semantics for analytic entailment from its definition $A \wedge B \leftrightarrow A$ in terms of analytic equivalence:

<u>Lemma 23</u> $[A \wedge B]^- = [A]^-$ iff $\lceil B \rceil^- \sqsubseteq \lceil A \rceil^- \;\&\; \lceil B \rceil^- \sqsupseteq \lceil A \rceil^-$
iff $[B]^- \subseteq [A]^-$.
Hence $A \wedge B \leftrightarrow A$ holds in the model both $[A]^+ > [B]^+$ and $[B]^- \subseteq [A]^-$.

<u>Proof</u> $[A \wedge B]^- = [A]^-$ iff $\lceil A \wedge B \rceil^- <> \lceil A \rceil^-$. Hence for the first result it suffices to show:

$\lceil A \wedge B \rceil^- <> \lceil A \rceil^-$ iff $\lceil B \rceil^- \sqsubseteq \lceil A \rceil^- \;\&\; \lceil B \rceil^- \sqsupseteq \lceil A \rceil^-$.

Suppose $\lceil A \wedge B \rceil^- <> \lceil A \rceil^-$. Then every (inclusive) falsifier for $A \wedge B$, and a falsifier for B, in particular, should contain and be contained in a falsifier for A. Now suppose $\lceil B \rceil^- \sqsubseteq \lceil A \rceil^- \;\&\; \lceil B \rceil^- \sqsupseteq \lceil A \rceil^-$. Then a falsifier for A will be a falsifier for $A \wedge B$, on the one hand, and a falsifier for $A \wedge B$, on the other hand, will either be (a) a falsifier for A or (b) a falsifier for B and hence containing and contained in a falsifier for A, or (c) the fusion $s = t \sqcup u$ of a falsifier t for A and a falsifier u for B and so s will contain a falsifier t for A and, since u is contained in a falsifier u^+ for A, s will also be contained in a falsifier $t \sqcup u^+$ for A.

We next show:

$\lceil B \rceil^- \sqsubseteq \lceil A \rceil^- \;\&\; \lceil B \rceil^- \sqsupseteq \lceil A \rceil^-$ iff $[B]^- \subseteq [A]^-$.

Suppose the condition on the left and take $u \in [B]^-$. Then $u^- \sqsubseteq u \sqsubseteq u^+$ for some (inclusive) falsifiers u^- and u^+ of B. But then u^- contains a falsifier t^- for A and u^+ is contained in a falsifier t^+ for A. So $t^- \sqsubseteq u \sqsubseteq t^+$ and, consequently, $u \in [A]^-$.

Now suppose $[B]^- \subseteq [A]^-$ and take a falsifier t of B. Then $t \in [B]^-$ and so $t \in [A]^-$, and so for some falsifiers falsifier t^- and t^+ for A, $t^- \sqsubseteq u \sqsubseteq t^+$, thereby showing that t contains and is contained in a falsifier for A.

Finally, let us note that $A \wedge B \leftrightarrow A$ holds in the model iff $[A \wedge B]^+ = [B]^+$ and $[A \wedge B]^- = [B]^-$. By lemmas 12 and 13, $[A \wedge B]^+ = [B]^+$ iff $\lceil A \wedge B \rceil^+ > \lceil A \rceil^+$ and, by lemma 8, $\lceil A \wedge B \rceil^+ > \lceil A \rceil^+$ iff $[A \wedge B]^+ > [A]^+$.

Clearly, if $A \leftrightarrow B$ is valid under the bi-lateral semantics then it is valid under the uni-lateral semantics. It turns out that the converse also holds:

<u>Theorem 24</u> The system AC is sound for the bi-lateral semantics and hence every formula valid under the uni-lateral semantics is also valid under the bi-lateral semantics.

<u>Proof</u> The theorem $A \leftrightarrow B$ of AC will be valid under the bi-lateral semantics if $A \leftrightarrow B$ and $\neg A \leftrightarrow \neg B$ are both valid under the uni-lateral semantics. But this follows from the soundness of AC under the uni-lateral semantics (theorem 14) and the fact that NR is an admissiable rule (theorem 2).

<u>§9 A 4-Valued Semantics for AC</u>

We shall provide an alternative proof of completeness in terms of a combination of two four-valued matrices.

One is already familiar from the four-valued semantics for first-degree entailment. Let us use T and F for truth and falsity. The four truth-values of the semantics might then be signified by TF (true and false),T$\not{F}$ (exclusively true), F$\not{T}$ (exclusively false), and $\not{T}\not{F}$ (neither true nor false). Take a formula to be *true* (resp. *false*) under an assignment α of these four truth- values if its truth-value contains an uncrossed T (F). Truth-values are assigned to truth-functional formulas under a given assignment according to the following prescriptions:

(i) ¬A is true (false) iff A is false (true)

(ii) A ∧ B is true iff A and B are true, and
A ∧ B is false iff A or B is false;

(iii) A ∨ B is true iff A is true or B is true
A ∨ B is false iff both A and B are false.

We say C is a *first degree entailment of* A - A $|=_{fde}$ C - if, under any assignment C is true whenever A is true (in effect, TF and T$\not{F}$ are taken to be the designated values); and we say C is an *inexact consequence of* A - A $|=_i$ C - if, relative to any model, $\lceil A\rceil^+ \sqsupseteq \lceil C\rceil^+$. Within the present context, the notion of first degree entailment might be axiomatized by adding the axiom A → A ∨ B to $AC_{\rightarrow}$.

We can show that first degree entailment and inexact consequence coincide (a result first established, in effect, by van Fraassen [1969]):

<u>Theorem 25</u> A $\|\!-_i$ C iff A $|=_{fde}$ C.

<u>Proof</u> First suppose not A $|=_i$ C. Then for some model $\boldsymbol{M}$ and state *s* of $\boldsymbol{M}$, *s* (inexactly) verifies A but does not inexactly verify C. Define the assignment α by:

(i) p is true under α iff *s* inexactly verifies p, and

(ii) p is false under α iff *s* inexactly falsifies p.

An easy induction then shows that for any formula B:

(i)* B is true under α iff *s* inexactly verifies B, and

(ii)* B is false under α iff *s* inexactly falsifies B.

So A is true relative α and C is not and hence not A $|=_{fde}$ C.

Suppose now that not A $|=_{fde}$ C. So under some assignment α′, A is true and C is not true. Recall the definition of the canonical model $\boldsymbol{M}_c = (S_c, \sqsubseteq_c, |\cdot|_c)$ from the beginning of §7. Let $s_{\alpha'}$ ={p: p is true under α′} ∪ {¬p: p is false under α′}. We readily show:

(i)′ p is true under α′ iff $s_{\alpha'}$ inexactly verifies p (in the canonical model), and

(ii)′ p is false under α′ iff s_α inexactly falsifies p.

So from (i)* and (ii)* above, it follows for any formula B that:

(i)*′ B is true under α′ iff $s_{\alpha'}$ inexactly verifies B, and

(ii)*′ B is false relative to α′ iff s_α inexactly falsifies B.

Since A is true and C is not true under α′, s_α inexactly verifies A and fails to inexactly verify C and so not A $\|\!-_i$ C.

Analogous results may be established for the ‘reverse’ clause in the semantical account of partial entailment. In this case, we shall provide three alternative characterizations - one in terms of positive and negative occurrence, a second in terms of partial truth and falsity, and a third in terms of an axiomatic system.

The first is in terms of the valence, *positive* or *negative*, of the occurrence of a sentence letter in a formula. This is defined as follows:

(i) p occurs positively in p

(ii) if p occurs positively (negatively) in A then it occurs negatively (positively) in ¬A

(iii) if p occurs positively (negatively) in A or in B then it occurs positively (negatively) in A ∧ B and in A ∨ B.

Note that a sentence letter may occur both positively and negatively in a given formula or not occur at all and hence occur neither positively nor negatively.

Given the notions of positive and negative occurrence, we may say A *preserves the valence of* C if every sentence letter that occurs positively (negatively) in C occurs positively (negatively) in A.

We have the following axiom system AC_2 for the preservation of valence:

S1. A → ¬¬A
S2. ¬¬A → A
S3. A → (A * B)
S4. B → (A * B)
S5. A → B, B → C / A → C
S6 A → B / ¬A → ¬B
S7. A → C, B → C / (A * B) → C,

where * is used indifferently for ∧ and ∨. For later purposes, we should note that an easy induction establishes that if C ↔ A is a theorem of AC then C → A is a theorem of AC_2.

We can now establish the soundness and completeness of the system AC_2 with respect to valence:

Theorem 26 A preserves the valence of iff C → A is a theorem of AC_2.

Proof (Sketch) Soundness (the left to right direction) is established by an easy induction. For completeness, we first show:

(i) if A preserves the valence of p then p → A is a theorem

(ii) if A preserves the valence of ¬p then ¬p → A is a theorem

by induction on A. We then show:

(iii) if A preserves the valence of C then C → A is a theorem

by induction on C.

The third alternative characterization is in terms of the notions of partial truth (t) and partial falsity (f). As with 'complete' truth (T) and falsehood (F), we may distinguish four truth-values, which we denote by tf (both partially true and partially false), $t\not{f}$ (partially true but not partially false), $f\not{t}$ (partially false but not partially true), $\not{t}\not{f}$ (neither partially true nor partially false). It should be understood that partial truth or falsehood is taken to be relative to a state. Thus a statement may be neither partially true nor partially false since there may be no information in the state bearing upon its truth and falsity and a statement that is partly true but not partly false may still be false simpliciter since the failure to be partially false may simply arise from a lack of information in the state.

Truth-values are assigned to truth-functional formulas under a given assignment according to the following prescriptions:

PT(i) ¬A is partially true (false) iff A is partially false (true)

PT(ii) A ∧ B is partially true iff A is partially true or B is partially true, and
A ∧ B is partially false iff A is partially false or B is partially false;

PT(iii) A ∨ B is partially true iff A is partially true or B is partially true
A ∨ B is partially false iff A is partially false or B is partially false.

(Truth-tables along these lines have been considered by Humberstone [2003]). We say a formula is *partially true* (*partially false*) under an assignment α if its value under the assignment contains a t (contains an f) (this is analogous to the earlier stipulation for when a formula is true or is false). Under a slight abuse of notation, we shall sometimes write α(A) = t (or α(A) = f) if A is partially true (partially false) under α (note that both α(A) = t and α(A) = f can hold).

Given the notions of partial truth and falsehood, we say A *preserves the partial truth of* C if A is partially true under any assignment of partial truth-values under which C is partially true.

Partial truth and valence are connected as follows:

<u>Lemma 27</u> For α an assignment of partial truth-values,

(i) α(A) = t iff α(p) = t for some positively occurring sentence letter p in A or
α(p) = f for some negatively occurring sentence letter p in A;

(ii) α(A) = f iff α(p) = f for some positively occurring sentence letter p in A or
α(p) = t for some negatively occurring sentence letter p in A.

<u>Proof</u> By a straightforward induction.

We may now establish the equivalence of the preservation of partial truth and valence:

<u>Lemma 28</u> A preserves the partial truth of C iff A preserves the valence of C.

<u>Proof</u> Suppose that A does not preserve the valence of C. Then some sentence letter p occurs positively in C but not positively in A or negatively in C but not negatively in A. Without loss of generality, assume the former. Let α be such that:

α(p) = tf̸ and

α(q) = t̸f̸ for any sentence letter q not identical to p.

By lemma 27, α(C) = t but not α(A) = t since otherwise p would occur positively in A.

Now suppose A does not preserve the partial truth of C. Then for some partial truth-value assignment α, α(C) = t but not α(A) = t. By lemma 27, α(A) = t for some positively occurring sentence letter p in C or α(A) = f for some negatively occurring sentence letter p in C. Without loss of generality, suppose the former. By lemma 27 again, p does not occur positively in A; and so A does not preserve the valence of C.

The notions of partial truth and falsehood may also be connected in a natural way with truthmakers. Given a model ***M*** and a state *s*, say that *s exactly partially verifies* (*falsifies*) A if for some non-null $s^- \sqsubseteq s$, s^- exactly verifies A; and say that *s* (*inexactly*) *partially verifies* (*falsifies*) A if for some $s^+ \sqsupseteq s$, s^+ exactly partially verifies A. Alternatively, we may say that *s partially verifies* (*falsifies*) A if *s* overlaps with an exact verifier (falsifier) for A.

To establish the connection, we require an additional condition on a truthmaker model ***M***:

<u>Overlap</u> If *t* overlaps with the fusion of ⊔*T* in ***M*** then it overlaps with some member of *T*.

We then have:

<u>Lemma 29</u> Suppose the model ***M*** conforms to Overlap. Then the notions of partial verification and falsification at a state conform to clauses PT(i) - (iii) above.

<u>Proof</u> (i) ¬A is partially true (false) at *s* iff *s* overlaps with a verifier (falsifier) for ¬A
iff *s* overlaps with a falsifier (verifier) for A
iff A is partially false (true) at *s*

(ii) Suppose A ∧ B is partially true at *s*. Then *s* overlaps with a verifier *t* for A ∧ B. The verifier *t* is of the form $t_1 \sqcup t_2$, where t_1 is a verifier for A and t_2 a verifier for B; and so by Overlap, *s* overlaps with t_1 or t_2 and hence with a verifier for A or B.

Now suppose A is partially true at *s* (the case in which B is partially true is similar) Then *s* overlaps with a verifier *t* for A. But B has a verifier *u* and so *s* overlaps with a verifier $t \sqcup u$ for A ∧ B and so A ∧ B is partially true at *s*.

Next suppose A ∧ B is partially false at *s*. Then *s* overlaps with a falsifier *t* for A ∧ B. Without loss of generality, we can assume that *t* is of the form $t_1 \sqcup t_2$, where t_1 is a falsifier of A and t_2 a falsifier of B. By Overlap, *s* overlaps with t_1 or with t_2 and so A or B is partially false at *s*.

(iii) Analogous to (ii).

We now establish:

<u>Theorem 30</u> The following four conditions are equivalent:

(i) $|C| \sqsubseteq |A|$ for any model ***M***

(ii) C → A is a theorem of AC_2

(iii) A preserves the valence of C

(iv) A preserves the partial truth of C

<u>Proof</u> In the light of the previous results, it suffices to establish the equivalence of (i) and (ii). (ii) ⇒ (i) is established by an easy induction. To establish (i) ⇒ (ii), suppose $|C| \sqsubseteq |A|$ for any model ***M***. Then, in particular, $|C|_c \sqsubseteq |A|_c$ (where the content |•| is defined relative to the canonical model $\boldsymbol{M}_c$). C is provably equivalent in AC to a standard maximal disjunctive form $C_1 \vee C_2 \vee ... \vee C_m$ and A to a standard maximal disjunctive form $A_1 \vee A_2 \vee ... \vee A_n$, i.e. $C \leftrightarrow C_1 \vee C_2 \vee ... \vee C_m$ and $A_1 \vee A_2 \vee ... \vee A_n \leftrightarrow A$ are theorems of AC and so $C \rightarrow C_1 \vee C_2 \vee ... \vee C_m$ and $A_1 \vee A_2 \vee ... \vee A_n \rightarrow A$ are theorems of AC_2 by lemma. Since $|C|_c \sqsubseteq |A|_c$, $|C_1 \vee C_2 \vee ... \vee C_m|_c \sqsubseteq |A_1 \vee A_2 \vee ... \vee A_n|_c$. But then each $L(C_k)$ is included in some $L(A_l)$ by lemma 18, from which it readily follows that $C_1 \vee C_2 \vee ... \vee C_m \rightarrow A_1 \vee A_2 \vee ... \vee A_n$ is also a theorem of AC_2. And from these chain of implications, the theoremhood of C → A is easily seen to follow.

Angell [1989], p. 126, notes that (iii) is a necessary condition for A → C (or, rather, A ↔ (A ∧ C)) to be a theorem of AC and we see that, in combination with C being a first degree entailment from A, it is both a necessary and a sufficient condition..

First degree entailment amounts to the preservation of truth under the four valued semantics whose truth values are TF, T*F*, F*T*, and *TF*. From theorems 25 and 30 we therefore have the following result:

<u>Corollary 31</u> A → C is a theorem of AC iff C preserves the truth of A and A the partial truth of C.

We can use this result to provide a finite 16-valued matrix for AC. The sixteen values are the cross product of {TF, T*F*, F*T*, *TF*} and {tf, t*f*, f*t*, *tf*} and the designated values are the cross product of {TF, T*F*} and {f*t*, *tf*} (the preservation of partial truth from C to A is converted to the preservation of non- partial-truth from A to C). The 16 values can, in fact, be reduced to 7 since we may just differentiate one designated value, TF or T*F*, into four values, when paired with the values tf, t*f*, f*t* or *tf*. I do not know whether the number of values can be reduced further.

<u>§10 Extensions of AC</u>

I wish briefly to discuss some extensions of the system AC at both the syntactic and the semantic level

(1) We may allow ourselves to form truth-functional compounds of equivalence formulas A ↔ B and the sentence letters. Thus each equivalence A ↔ B and each sentence letter p is to

be a formula of the extended language; and if A and B are formulas of the extended language then so are ¬A, (A ∧ B) and (A ∨ B).

In extending the semantics, we may enrich each model with a set of actual or real states. Thus a model ***M*** is now an ordered quadruple (*S*, *R*, ⊑, |·|), where (*S*, ⊑, |·|) is a model in the previous sense and *R* is a subset of *S* subject to the following three conditions:

(a) the fusion of any subset of *R* is a member of *R*;

(b) the part of any member of *R* is a member of *R*;

(c) exactly one of $|p|^+$ or $|p|^-$ contains a member of *R* for each sentence letter p.

We define *s* ||- A and *s* -|| A for propositional formulas in the usual way relative to the model ***M*** (for which no appeal to the subset *R* is required). We then define what it is for a formula of the extended language to *hold* (*relative to* the model ***M***):

(i) A ↔ B holds iff $[A]^+ = [B]^+$ (under the uni-lateral semantics) or
iff $[A]^+ = [B]^+$ and $[A]^- = [B]^-$ (under the bi-lateral semantics)

(ii) the sentence letter p holds iff some element of *R* belongs to $|p|^+$;

(iii) ¬A holds iff A does not hold;

(iv) A ∧ B holds iff A holds and B holds;

(v) A ∨ B holds iff A holds or B holds.

The validity of formulas and inference rules is then defined in the usual way.[3]

An obvious way to get an axiom system for the uni-lateral semantics is to conditionalize the rules of our original system, add (A ↔ B) ⊃ (A ≡ B) as an axiom, and modus ponens as a rule of inference. We thereby obtain the following axioms and rules for the system AC*:

E0* A ↔ B ⊃ (A ⊃ B)
E1 A ↔ ¬¬A
E2 A ↔ A ∧ A
E3 A ∧ B ↔ B ∧ A
E4 (A ∧ B) ∧ C ↔ A ∧ (B ∧ C)
E5 A ↔ A ∨ A
E6 A ∨ B ↔ B ∨ A
E7 (A ∨ B) ∨ C ↔ A ∨ (B ∨ C)
E8 ¬(A ∧ B) ↔ (¬A ∨ ¬B)
E9 ¬(A ∨ B) ↔ (¬A ∧ ¬B)
E10 A ∧ (B ∨ C) ↔ (A ∧ B) ∨ (A ∧ C)
E11 A ∨ (B ∧ C) ↔ (A ∨ B) ∧ (A ∨ C)
E12* A ↔ B ⊃ B ↔ A
E13* A ↔ B ⊃ (B ↔ C ⊃ A ↔ C)
E14* A ↔ B ⊃ (A ∧ C) ↔ (B ∧ C)
E15* A ↔ B ⊃ (A ∨ C) ↔ (B ∨ C)
E16* $A_1, A_2, ..., A_n$/B whenever B is a truth-functional consequence of $A_1, A_2, ..., A_n$.

It is readily shown that each theorem of AC is also a theorem of AC* and that AC* is sound for the extended semantics. Under the bi-lateral semantics, we should also A ↔ B ⊃ ¬A ↔ ¬B as an axiom. I do not have a proof of completeness in either case.

(2) We might also allow iterated applications of ↔. Thus all sentence letters are to be formulas and ¬A, (A ∧ B), (A ∨ B) and (A ↔ B) are to be formulas whenever A and B are

[3]Models with a component *R* for 'reality' are also considered in Fine [2013].

formulas. To extend the semantics to the new system, we would need to say when a state verified or falsified a formula of the form (A ↔ B). Let ∧ (the fusion of no states) and ∨ (the fusion of all states) be the bottom and top elements of an extended model ***M*** = (*S*, *R*, ⊑, |·|). Then the simplest rule for ↔ is:

s ‖- A ↔ B if [A] = [B] and *s* = ∧;
s -‖ A ↔ B if [A] ≠ [B] and *s* = ∨.

But this rule would result in the analytic equivalence of all true analytic equivalences and of all false analytic equivalences; and it would be preferable to have a more refined rule in which the the verifying or falsifying state somehow reflected how the equivalence was true or false.

(3) One might extend the system and the semantics to the quantifiers. The obvious way to do this is to treat a universal quantified statement as a conjunction of its instances and an existential quantified statement as the disjunction of its instances. But there are more refined treatments of the quantifiers that might also be considered.

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